KINGDOM OF SAUDI ARABIA
Northern Border University (NBU)
Journal of the North for Basic & Applied Sciences(JNBAS)
p - ISSN : 1658 -7022 / **e-ISSN:** 1658 - 7014
www.nbu.edu.sa
s.journal@nbu.edu.sa

Classification of Complemented Subspaces in $C_0(\omega^\omega \times \mathcal{K}_{\mathcal{A}})$ for Spaces with Few Operators

Mortada S. Ali
Department of Mathematics, Faculty of Sciences Al-Baha University, P.O. Box-7738, Alaqiq, Al-Baha-65799, Saudi Arabia.

(**Received:** 24th April 2026; **Accepted:** 7th June 2026)

Abstract
Building on Acuaviva's work, which characterizes operators on injective tensor products (Acuaviva, 2026), we extend the classification of complemented subspaces from $C_0(\omega \times \mathcal{K}_{\mathcal{A}})$ to $C_0(\omega^\omega \times \mathcal{K}_{\mathcal{A}})$, where $\mathcal{K}_{\mathcal{A}}$ is generated by an almost disjoint family $\mathcal{A}$ of continuum cardinality with few operators. In addition, we adapt Benyamini's extension theorem for separable spaces (Benyamini,1978) to show that any infinite dimensional complemented subspace is isomorphic to C_0, $C_0(\mathcal{K}_{\mathcal{A}})^n$ for $n \in \mathbb{N}$, or $C_0(\omega^\omega \times \mathcal{K}_{\mathcal{A}})$. We use inductive arguments on Cantor-Bendixson indices and tensor decompositions, and we confirm uniqueness follows from the structural decomposition and separable-range analysis. This addresses a technical gap in handling transfinite ordinals exceeding ω, with applications to lattices of operator ideal and potential generalizations to non-scattered spaces.

Keywords: Injective tensor products, operator ideals, complemented subspaces, few operators, transfinite ordinals, scattered compact spaces.

1658-7022© JNBAS. (1448 H/2026). Published by Northern Border University (NBU). All Rights Reserved.

(*) **Corresponding Author**:
Mortada S. Ali
Department of Mathematics, Faculty of Sciences Al-Baha University, P.O. Box-7738, Alaqiq, Al-Baha-65799, Saudi Arabia.

E-mail: mortada@bu.edu.sa

DOI: 10.12816/0062627

1. Introduction

The study of Banach spaces with restricted operator structures has been a central theme in modern functional analysis. A Banach space $\mathcal{X}$ is said to have few operators if every bounded linear operator $\mathcal{T}:\mathcal{X}\to\mathcal{X}$ admits a decomposition $\mathcal{T}=\lambda I+\mathcal{S}$, where $\lambda\in\mathbb{k}$ is a scalar and $S:\mathcal{X}\to\mathcal{X}$ is an operator with separable range. This structural rigidity, first established in ZFC-independent contexts by Shelah (1978) and later refined by Shelah & Steprāns (1988), reduces the complexity of the operator algebra $B(\mathcal{X})$ significantly: the non-scalar part is restricted to operators with separable range.

On a separate track, Benyamini (1978) showed that every infinite-dimensional complemented subspace of $C(\omega^\omega)$ is isomorphic to either c_0 or the entire space $C(\omega^\omega)$, relying on an extension theorem designed for separable scattered compacta. In parallel, Candido (2021) analyzed spaces of the form $C_0(\alpha\times L)$, demonstrating that finite-power complements $C_0(L)^n$ can arise under the auxiliary set-theoretic ♣-principle. Later, Acuaviva (2026) studied tensor-product structures for spaces of the form $C_0(\omega \times \mathcal{K}_{\mathcal{A}})$. Extending this work transfinitely to $C_0(\omega^\omega\times\mathcal{K}_{\mathcal{A}})$, however, poses new challenges specifically, one needs finer control over separable-range perturbations at limit ordinal stages. The present paper tackles this setting within ZFC, combining transfinite decomposition techniques with the few-operator framework. This extends Candido's finite-power results to the transfinite ordinal ω^ω while eliminating the dependence on combinatorial axioms.

The main result is established using a transfinite induction framework together with the operator decomposition methods developed in Section 3:

Theorem 1.1.

Assume that $\mathcal{K}_{\mathcal{A}}$ is a locally compact Hausdorff space, with the generating almost disjoint family $\mathcal{A}\subseteq[\mathbb{N}]^\omega$ of continuum cardinality, and that the function space $C_0(\mathcal{K}_{\mathcal{A}})$ admits only few operators. Then, every infinite dimensional complemented subspace of $C_0(\omega^\omega\times\mathcal{K}_{\mathcal{A}})$ must be isomorphic to one of the following spaces:c_0, $C_0(\mathcal{K}_{\mathcal{A}})^n$ for some finite $n\in\mathbb{N}$; or $C_0(\omega^\omega\times\mathcal{K}_{\mathcal{A}})$.

This resolves the question raised in (Acuaviva, 2026) concerning transfinite extension beyond ω. We also discuss applications to operator ideals and indicate possible broader extensions, motivated by more developments, including. Candido's extensions of $C_0(L\times L)$,(Candido,2004).

2. Preliminaries

We adhere to standard notation used in Banach space theory. All spaces are taken over field $\mathbb{K}$, where $\mathbb{k}\in\{\mathbb{R},\mathbb{C}\}$. An operator always refers to a bounded linear map between normed spaces. For Banach space $\mathcal{X}$, we denote its unit ball by $B_{\mathcal{X}}$. For Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, we write $\mathcal{X}\sim=\mathcal{Y}$ if they are (linearly) isomorphic, i.e., there exists a bounded linear bijection $\mathcal{T}:\mathcal{X}\to\mathcal{Y}$ with bounded inverse. We write $\mathcal{X}\sim\mathcal{Y}$ if they are isometrically isomorphic, i.e., there exists a linear bijection $\mathcal{T}:\mathcal{X}\to\mathcal{Y}$ such that $\|\mathcal{T}\mathcal{X}\|=\|\mathcal{X}\|$ for all $\mathcal{X}\in\mathcal{X}$. Additional specialized notation will be given when necessary.

2.1 Tensor Products and Few Operators

We give a brief overview of well-known results regarding tensor products of Banach spaces, focusing particularly on the injective tensor product, as discussed in (Ryan, 2002). We will assume standard knowledge of the algebraic tensor product, denoted $\mathcal{X}\otimes\mathcal{Y}$, of vector spaces $\mathcal{X}$ and $\mathcal{Y}$; see Ryan (2002, Chapter 1) again for the elementary definitions. Note that if E,F,$\mathcal{X}$ and $\mathcal{Y}$ are vector spaces and U:E→F,R:$\mathcal{X}\to\mathcal{Y}$ are linear maps, the tensor product of U and R is a linear map

$$U\otimes R: E\otimes\mathcal{X}\to F\otimes\mathcal{Y}$$

that satisfies the usual properties:

$$(U\otimes R)(e\otimes x)=(Ue)\otimes(Rx),\ \ \forall\, e\in E \text{ and } x\in\mathcal{X}.$$

Let $\mathcal{X}$ and $\mathcal{Y}$ be two Banach spaces. For these spaces, we may consider their algebraic tensor product $\mathcal{X}\otimes\mathcal{Y}$, that is, the vector space consisting of finite sums of the form $\mathcal{X}\otimes\mathcal{Y}$, with $x\in\mathcal{X}$ and y$\in\mathcal{Y}$. The algebraic tensor product can be equipped with different norms, and different norms may yield different Banach spaces upon completing the spaces with respect to the norms. The theory of tensor products of Banach spaces investigates the definition of tensor product norms and the resulting properties of the completed Banach spaces.

If $\alpha(\cdot)$ is a norm defined on the algebraic tensor product $\mathcal{X}\otimes\mathcal{Y}$. The completion of $\mathcal{X}\otimes\mathcal{Y}$ with respect to α is denoted by $\mathcal{X}\hat{\otimes}_\alpha\mathcal{Y}$. Two of the most commonly used norms are the projective norm $\pi(\cdot)$ and the injective norm $\varepsilon(\cdot)$ (see Chapters 2 and 3 in Ryan,2002). If an element $u\in\mathcal{X}\otimes\mathcal{Y}$ can be written as

$$u=\sum_{i=1}^{n} x_i\otimes y_i,$$

then it follows that

$$\varepsilon(u)=\sup\left\{\left|\sum_{i=1}^{n}\varphi(x_i)\,\psi(y_i)\right| : \varphi\in B_{\mathcal{X}^*},\psi\in B_{\mathcal{Y}^*}\right\},$$

and this norm is well-defined and does not depend on the particular representation chosen for u. Another fundamental property is

$$\varepsilon(x\otimes y)=\|x\|\|y\| \quad \text{for each } x\in\mathcal{X} \text{ and } y\in\mathcal{Y},$$

(see Proposition 3.1 in Ryan, 2002).

A key fact that will be used later is that the injective tensor product preserves subspaces. Moreover, we have the following result:

Proposition 2.1

If U:E→F and R:$\mathcal{X}\to\mathcal{Y}$ are linear operators. Then there exists a unique operator

$$U \widehat{\otimes} R: E \widehat{\otimes}_{\varepsilon} \mathcal{X} \to F \widehat{\otimes}_{\varepsilon} \mathcal{Y}$$

such that

$$(U \widehat{\otimes} R)(e \otimes x) = (Ue) \otimes (Rx), \text{ for all } e \in E \text{ and } x \in \mathcal{X}.$$

Moreover, the operator norm satisfies

$$\| U \widehat{\otimes} R \| = \| U \| \| R \|.$$

Observe that, U$\widehat{\otimes}$R is obtained as the continuous extension of the linear map U⊗R:E⊗$\mathcal{X}$→F⊗$\mathcal{Y}$. Additionally, recall the following standard result. (see Section 3.2 in Ryan, 2002).

Proposition 2.2

Let $\mathcal{X}$ be a Banach space, and K and L be two compact Hausdorff spaces. Then we have the canonical isometric isomorphism

$$C(K) \widehat{\otimes}_{\varepsilon} \mathcal{X} \sim= C(K, \mathcal{X}),$$

where elementary tensors are identified via

$$(f \otimes x)(k) = f(k)x \quad \text{for all} \quad f \in C(K), x \in \mathcal{X}, \quad \text{and } k \in K.$$

Likewise, we obtain the isometric isomorphism

$$C(K) \widehat{\otimes}_{\varepsilon} C(L) \sim= C(K \times L).$$

As observed above, the injective tensor product satisfies the following:

If U:E→F and R:$\mathcal{X}\to\mathcal{Y}$ are two bounded operators, then the algebraic tensor U⊗R extends to a bounded operator

$$U \widehat{\otimes} R: E \widehat{\otimes}_{\varepsilon} \mathcal{X} \to F \widehat{\otimes}_{\varepsilon} \mathcal{Y},$$

It is natural to ask for which norms α(·) on the algebraic tensor product such an extension exists; that is, for which tensor norms the linear map U⊗R extends to a bounded operator

$$U \widehat{\otimes} R: E \widehat{\otimes}_{\alpha} \mathcal{X} \to F \widehat{\otimes}_{\varepsilon} \mathcal{Y},$$

where we highlight the fact that the map is defined from the α-tensor product E$\widehat{\otimes}_{\alpha}\mathcal{X}$ into the injective tensor product F$\widehat{\otimes}_{\varepsilon}\mathcal{Y}$.

The present paper takes a closer look at certain well-behaved norms (see Section 6.2 in Ryan, 2002), which, in particular ensure the existence of such operator extensions.

Definition (Ryan,2002) 2.3

Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. A reasonable cross-norm α(·) on $\mathcal{X}\otimes\mathcal{Y}$ is a norm that satisfies:

(1) $\alpha(x \otimes y) \le \|x\|\|y\|, \ \forall\, x \in \mathcal{X}$ and $y \in \mathcal{Y}$,

(2) For every bounded linear functional φ in $\mathcal{X}^*$ and ψ in $\mathcal{Y}^*$, the elementary tensor φ⊗ψ , regarded as a linear functional on $\mathcal{X}\otimes$Y, is bounded, and moreover

$$\|\varphi \otimes \psi\| \le \|\varphi\|\|\psi\|.$$

Definition (Ryan,2002) 2.4

A uniform cross-norm α(·) is an assignment of a reasonable cross-norm to each pair of Banach spaces (E,$\mathcal{X}$), such that for every bounded linear operator U:E→F and R:$\mathcal{X}\to\mathcal{Y}$, the tensor product operator U⊗R:E⊗$\mathcal{X}$→F⊗$\mathcal{Y}$ satisfies

$$\|U \otimes R\| \le \|U\|\|R\|,$$

where ‖U⊗R‖ is taken with respect to the cross-norms on the spaces E⊗$\mathcal{X}$ and F⊗$\mathcal{Y}$.

It is a fact that α(·) is a reasonable cross-norm if, and only if,

$$\varepsilon(u) \le \alpha(u) \le \pi(u), \forall\, u \in \mathcal{X} \otimes \mathcal{Y},$$

as demonstrated in Proposition 6.1 of Ryan (2002). This implies that, in case α(·) is a uniform crossnorm and U:E→F and R:$\mathcal{X}\to\mathcal{Y}$ are operators, the product operator

$$U \widehat{\otimes} R: E \widehat{\otimes}_{\alpha} \mathcal{X} \to F \widehat{\otimes}_{\varepsilon} \mathcal{Y},$$

is well-defined, satisfies that

$$\|U \widehat{\otimes} R\| \le \|U\|\|R\|$$

and acts on elementary tensors as

$$(U \widehat{\otimes} R)(e \otimes x) = (U \otimes R)(e \otimes x) = (Ue) \otimes (Rx)$$

for each e∈E and x∈X.

Finally, we observe that ε(·) is itself a uniform cross-norm. That property is required for the purposes of the theory of tensor products of Banach spaces developed here.

A space $\mathcal{X}$ is said to have only a few operators, if all operators $\mathcal{T}$:$\mathcal{X}\to\mathcal{X}$ are of the form $\mathcal{T}$=λI+$\mathcal{S}$ where λ is a scalar in the field $\mathbb{K}$ and $\mathcal{S}$ has separable range (Shelah, 1978).

According to the findings presented in (Acuaviva, 2026): If E and F are separable spaces, and $\mathcal{X}$ is a non-separable space with few operators,

$$\mathcal{T}: E \widehat{\otimes}_\alpha \mathcal{X} \to F \widehat{\otimes}_\varepsilon \mathcal{X},$$

can be represented as $\mathcal{T}=U\widehat{\otimes}I+\mathcal{S}$, where $U:E\to F$ is a bounded operator and $\mathcal{S}$ is an operator, whose range is separable. This theorem utilizes the elementary structure of operators on $\mathcal{X}$, establishing a connection to Shelah's constructions within the framework of the diamond axiom (which subsequently became ZFC-independent (Shelah & Steprāns, 1988)) and the reflexive examples presented in (Wark, 2001).

Standardization of Notation. To ensure consistency throughout the manuscript, we fix the following conventions. Let $\mathcal{X}$ be a Banach space. We denote by $B(\mathcal{X})$ the algebra of bounded linear operators, by $K(\mathcal{X})$ the closed ideal of compact operators, by $\mathcal{S}(\mathcal{X})$ the closed ideal of operators with separable range, and by $G_E(\mathcal{X})$ the ideal of operators factoring through a Banach space E. The symbol $\bigoplus_\infty$ denotes the ℓ_∞-direct sum equipped with the maximum norm. For compact spaces K we write $C(K)$, and for locally compact non-compact spaces L we write $C_0(L)$. Throughout this work, the phrase "$\mathcal{X}$ has few operators" strictly means that every $\mathcal{T}\in B(\mathcal{X})$ decomposes as $\mathcal{T}=\lambda I+\mathcal{S}$ with $\lambda\in\mathbb{K}$ and $\mathcal{S}\in\mathcal{S}(\mathcal{X})$ (Shelah, 1978). All subsequent references to operator ideals, tensor products, and ordinal derivatives follow this standardized notation

2.2 Almost Disjoint Families and $C_0(\mathcal{K}_\mathcal{A})$

We use $[\mathbb{N}]^{<\omega}$ and $[\mathbb{N}]^{<\omega}$ to denote the families of subsets of $\mathbb{N}$, finite or infinite respectively. A subfamily $\mathcal{A}\subseteq[\mathbb{N}]^\omega$ is called almost disjoint if the intersection of any two distinct elements $A,B\in\mathcal{A}$, is finite; that is, $A\cap B\in[\mathbb{N}]^{<\omega}$ (Hjorth, 2005).

In this section, we follow the notation presented in (Koszmider & Laustsen, 2021) and denote by $\mathcal{K}_\mathcal{A}$ the locally compact Hausdorff space corresponding to the almost disjoint collection $\mathcal{A}\subseteq[\mathbb{N}]^\omega$, as given in Definition 3 of Koszmider and Laustsen (2021). For convenience, we recall the following definition:

Definition 2.5

(Koszmider & Laustsen, 2021) Assume that $\mathcal{A}\subseteq[N]^\omega$ is an almost disjoint family. The space $\mathcal{K}_\mathcal{A}$ is defined as the set of points

$$\{x_n : n\in\mathbb{N}\}\cup\{y_A : A\in\mathcal{A}\},$$

where each x_n is an isolated point.

For each $A\in\mathcal{A}$ and finite subset $G\subseteq\mathbb{N}$, we define the sets

$$U(A,G)=\{x_n : n\in A\setminus G\}\cup\{y_A\},$$

These sets form a neighbourhood basis at the point y_A. We also denote $U(A)$ simply by $U(A,\varphi)$.

We first summarize key properties of these spaces (see Koszmider & Laustsen, 2021) and extend certain results from the same work to their natural analogues in $C_0(M\times\mathcal{K}_\mathcal{A})$ where M is a compact metric space.

Lemma 2.6 Suppose that $\mathcal{A}\subseteq[\mathbb{N}]^\omega$ is an almost disjoint family. Then:

- $\mathcal{K}_\mathcal{A}$ is a locally compact, scattered and Hausdorff space.
- Compactness of $\mathcal{K}_\mathcal{A}$ occurs if and only if both $\mathcal{A}$ and $\mathbb{N}\setminus\bigcup\mathcal{A}$ are finite.
- The set $\{\mathcal{X}_n : n\in\mathbb{N}\}$, which consists of isolated points of $\mathcal{K}_\mathcal{A}$, is dense in $\mathcal{K}_\mathcal{A}$ so $\mathcal{K}_\mathcal{A}$ is separable. Consequently $\mathcal{K}_\mathcal{A}$ is metrizable if and only if it is second countable, which occurs precisely when $\mathcal{A}$ is countable.
- The family of points $\{y_A : A\in\mathcal{A}\}$ forms a closed and discrete subspace of $\mathcal{K}_\mathcal{A}\setminus\{x_n : n\in\mathbb{N}\}$.
- For every $A\in\mathcal{A}$, the sequence $(x_n)_{n\in A}$ converges to y_A in $\mathcal{K}_\mathcal{A}$.

Definition 2.7 Suppose that $\mathcal{A}\subseteq[\mathbb{N}]^\omega$ is an almost disjoint family. For a function $f\in C_0(M\times\mathcal{K}_\mathcal{A})$ and a set of functions $X\subseteq C_0(M\times\mathcal{K}_\mathcal{A})$, we define the following support sets

1. $s(f)=\{A\in\mathcal{A}:$ there exists $k\in M$ with $f(k,y_A)\neq 0\}$
2. $s(X)=\{s(f): f\in X\}$, where $s(f)$ is defined as above.

Lemma 2.8 For every function $f\in C_0(M\times\mathcal{K}_\mathcal{A})$, the set

$$s(f)=\{A\in\mathcal{A}:\exists k\in M \text{ such that } f(k,y_A)\neq 0\}$$

is countable.

Proof. We fix a countable dense subset $M_0\subseteq M$. We claim that

$$s(f)=\{A\in\mathcal{A}:\exists k\in M_0 \text{ such that } f(k,y_A)\neq 0\}=\bigcup_{k\in M_0}\{A\in\mathcal{A}: f(k,y_A)\neq 0\}.$$

Indeed, suppose $A\in s(f)$. By definition, there exists $k\in M$ such that $f(k,y_A)\neq 0$. Since M_0 is dense in M, there exists a sequence $(k_n)\subset M_0$ with $k_n\to k$. By continuity of f in the first variable, we have

$$f(k_n,y_A)\to f(k,y_A)\neq 0.$$

Hence, there exists $N\in\mathbb{N}$ such that $f(k_N,y_A)\neq 0$. This shows that it suffices to consider only $k\in M_0$ when defining $\mathcal{S}(f)$.

Now, it remains to demonstrate that the set

$$\{A\in\mathcal{A}: f(k,y_A)\neq 0\}$$

is countable for all $k\in M_0$.

Fix $k \in M_0$. Observe that the set $\{(k,y_A): A \in \mathcal{A}\}$ is discrete and closed in $\mathcal{K}_{\mathcal{A}}$ In particular, only finite subsets of a discrete closed set can be compact.

On the other hand, for all $\varepsilon>0$, the set

$$\{(k, y_A) \in M \times \mathcal{K}_{\mathcal{A}} : |f(k, y_A)| \geq \varepsilon\}$$

is compact, since $f \in C_0(M \times \mathcal{K}_{\mathcal{A}})$.

Combining these observations, any subset of $\{(k,y_A): A \in \mathcal{A}\}$ on which $|f(k,y_A)| \geq \varepsilon$ is finite.

Finally, we can write

$$\{A \in \mathcal{A} : f(k, y_A) \neq 0\} = \bigcup_{n=1}^{\infty} \{A \in \mathcal{A} : |f(k, y_A)| \geq 1/n\}.$$

Each set on the right-hand side is finite, hence the union is countable. Since M_0 is countable and

$$s(f) = \bigcup_{k \in M_0} \{A \in \mathcal{A} : f(k, y_A) \neq 0\},$$

we conclude that $s(f)$ is countable.

Lemma. Suppose that

$$\mathcal{A} \subseteq [N]^{\omega},$$

is an almost disjoint family and let

$$X \subseteq C_0(M \times \mathcal{K}_{\mathcal{A}}),$$

be a closed subspace. Then: The space X is separable precisely when its support set s(X) is countable.

Lemma 2.9 Suppose

$$\mathcal{A} \subseteq [N]^{\omega},$$

is an almost disjoint family, and let

$$X \subseteq C_0(M \times \mathcal{K}_{\mathcal{A}}),$$

be a closed subspace. Then:

There exists $Y \subseteq C_0(M \times \mathcal{K}_{\mathcal{A}})$ such that

$$X \subseteq Y \text{ and } Y \sim C_0(M \times \omega).$$

More precisely:

1. If M is infinite, then
$$Y \sim C(M).$$
2. If M is finite, then
$$Y \sim C_0.$$

Lemma 2.10 Suppose that

$$\mathcal{P}: C_0(M \times \mathcal{K}_{\mathcal{A}}) \to C_0(M \times \mathcal{K}_{\mathcal{A}}),$$

be a projection, and let

$$\mathcal{P} = Q \widehat{\otimes} I + \mathcal{S},$$

where

3. $Q: C(M) \to C(M)$ is a bounded operator,
4. I is the identity operator on $C(\mathcal{K}_{\mathcal{A}})$,
5. $\mathcal{S}$ is an operator whose range is separable.

Then Q is also a projection.

Lemma 2.11 Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, and let

$$\mathcal{P}: \mathcal{X} \to \mathcal{X}$$

be a projection. If $\mathcal{P}$ factors through $\mathcal{Y}$; that is, there exist bounded operators

$$\mathcal{T}: \mathcal{X} \to \mathcal{Y}, \qquad \mathcal{S}: \mathcal{Y} \to \mathcal{X}$$

such that

$$\mathcal{P} = \mathcal{S} \circ \mathcal{T}$$

Then $\mathcal{P}[\mathcal{X}]$ is isomorphic to a complemented subspace of $\mathcal{Y}$.

An almost disjoint family $\mathcal{A} \subseteq [N]^{\omega}$ has finite intersections. $\mathcal{K}_{\mathcal{A}}$ is the space with points $\{x_n : n \in \mathbb{N}\} \cup \{y_A : A \in \mathcal{A}\}$, topology as in Definition 4.1 of Acuaviva (2026) or Definition 3 of Koszmider and Laustsen (2021). It is scattered, separable, and non-metrizable for uncountable $\mathcal{A}$ (Koszmider & Laustsen, 2021). Subspaces are separable if and only if their supports are countable (Acuaviva, 2026, Lemma 4.5), aligning with the theory that $C_0(K)$ for scattered K has bounded approximation property (Kania & Kochanek, 2014).

3. Proof of Theorem 1.1

We extend Theorem 1.4 of Acuaviva (2026) by applying induction on the Cantor-Bendixson index. Recall that ω^{ω} has index $\omega+1$, with derivatives $(\omega^{\omega})^{(n)} = \omega^{\omega-n}$ as stated in (Bessaga & Pełczyński,1960).

3.1 Operator Characterization for Higher Ordinals

First, we generalize Corollary 1.3 of Acuaviva (2026) to $C_0(\omega^{\omega} \times L)$.

Lemma 3.1. Let L be non-metrizable with $C_0(L)$ having few operators, and M_1, M_2 be countable ordinals. Any operator

$$\mathcal{T}: C_0(M_1 \times L) \to C_0(M_2 \times L),$$

can be represented in the form $\mathcal{T} = U \widehat{\otimes} I + \mathcal{S}$, with $U: C(M_1) \to C(M_2)$ and S having a separable range.

Proof.

We proceed by transfinite induction on the countable ordinal α representing M_1.

Base case ($\alpha<\omega$): If $M_1=n$ is finite, then $C_0(n\times L)\cong C_0(L)^n$ equipped with the ℓ_∞-norm. Any bounded operator $\mathcal{T}$: $C_0(L)^n \to C_0(M_2\times L)$ is uniquely determined by an n-tuple $(\mathcal{T}_1,\cdots,\mathcal{T}_n)$ where $\mathcal{T}_i$:$C_0(L)\to C_0(M_2\times L)$. Using the canonical isometric isomorphism $C_0(M_2\times L)\cong C(M_2)\ \widehat{\otimes}_\varepsilon C_0(L)$ and the structural characterization of operators on injective tensor products with few-operator factors (Acuaviva, 2026), each T_i admits a decomposition $\mathcal{T}_i=V_i\ \widehat{\otimes} I+\mathcal{S}_i$, where $V_i\in B(C_0(L),C(M_2))$ and $\mathcal{S}_i$ has separable range. Define U:$C(n)\to C(M_2)$ by mapping the standard basis of $C(n)$ to $\{V_1,\cdots,V_n\}$, and set $\mathcal{S}=\bigoplus^n_{i=1}\mathcal{S}_i$. Since a finite ℓ_∞-sum of separable subspaces is separable, ran($\mathcal{S}$) is separable. Hence $\mathcal{T}=U\otimes I+\mathcal{S}$, holds for finite ordinals

Successor step ($\alpha+1$): Assume the representation holds for a countable ordinal α. The order topology yields the isometric decomposition $C_0((\alpha+1)\times L)\cong C_0(\alpha\times L)\cong C_0(\alpha\times L)\oplus_\infty C_0(L)$, where the second summand corresponds to the isolated fiber $\{\alpha\}\times L$. Let $\mathcal{T}$:$C_0((\alpha+1)\times L)\to C_0(M_2\times L)$. Restricting T to the first summand and applying the induction hypothesis gives $\mathcal{T}|_{C0(\alpha\times L)}=U_\alpha\ \widehat{\otimes} I+\mathcal{S}_\alpha$. For the isolated fiber, define R: $C_0(L)\to C_0(M_2\times L)$ by $R(f)=\mathcal{T}(0\oplus f)$. By the operator decomposition theorem for injective tensor products with few-operator factors established in Acuaviva (2026, Theorem 1.1), applied to the canonical identification $C_0(M_2\times L)\cong C(M_2)\ \widehat{\otimes} C_0(L)$, the operator R admits a decomposition $R=V\ \widehat{\otimes} I+\mathcal{S}_R$, where $V\in B(C_0(L),C(M_2))$ and $\mathcal{S}_R$ has separable range. Define U:$C(\alpha+1)\to C(M_2)$ by extending U_α and assigning V to the isolated coordinate. Set $\mathcal{S}=S_\alpha\oplus\mathcal{S}_R$. The range of $\mathcal{S}$ remains separable as a direct sum of two separable subspaces. Thus $\mathcal{T}=U\ \widehat{\otimes} I+\mathcal{S}$ holds for $\alpha+1$.

Limit step (λ): Let λ be a countable limit ordinal. Then $C_0(\lambda\widehat{\times}L)=(\bigcup_{(\gamma<\lambda)}C_0(\gamma\times\lambda)$, where the union is directed under inclusion. For each $\gamma<\lambda$, the restriction $\mathcal{T}_\gamma=\mathcal{T}|_{(C0\ (\gamma\times L)}$ satisfies the induction hypothesis: $\mathcal{T}_\gamma=U_\gamma\ \widehat{\otimes} I+S_\gamma$. The family (U_γ) is consistent under restriction $U_\delta|_{C0\ (\gamma\times L)}=U_\gamma$ for $\gamma<\delta<\lambda$) and uniformly bounded by $\|\mathcal{T}\|$. For $f\in\bigcup_{(\gamma<\lambda)}C_0(\gamma\times L)$, choose $\gamma<\lambda$ such that $f\in C_0(\gamma\times L)$, and define $U(f)=U_\gamma(f)$. This is well defined by the restriction consistency above. Hence, (U_γ) defines a bounded linear operator U on the dense subspace $\bigcup_{(\gamma<\lambda)}C_0(\gamma\times\lambda)$, which extends uniquely by continuity to U:$C(\lambda)\to C(M_2)$. Define $\mathcal{S}$ on the dense union by $\mathcal{S}|_{C0\ (\gamma\times L)}=\mathcal{S}_\gamma$. Since $\mathcal{T}_\delta|_{C0\ (\gamma\times L)}=\mathcal{T}_\gamma$ and $U_\delta|_{C0\ (\gamma\times L)}=U_\gamma(\gamma<\delta<\lambda)$, it follows that $\mathcal{S}_\delta|_{C0\ (\gamma\times L)}=\mathcal{S}_\gamma$, so $\mathcal{S}$ is also well defined. Each ran $(\mathcal{S}_\gamma)$ is separable. Since λ is countable, ran$(\mathcal{S})\subseteq\bigcup_{\gamma<\lambda}$ ran $\mathcal{S}_\gamma$, and the latter is a separable closed subspace, being the closure of a countable union of separable subsets of a metric space.

The uniform boundedness of the extensions and the continuity of the injective tensor norm guarantee that $\mathcal{T}=U\ \widehat{\otimes} I+\mathcal{S}$, on the dense union; equivalently, for every $Y<\lambda$, $\mathcal{T}|_{c0\ (\gamma\times L)}=U_\gamma\ \widehat{\otimes} I+\mathcal{S}_\gamma$. Therefore, by density and continuity of the three bounded operators, the identity extends to the completion C0 ($\lambda\times L$).This completes the induction.

Remark 3.2. For $\alpha=\omega^\omega$, the separable perturbation $\mathcal{S}$ factors through $C_0(\Gamma)$ for some countable Γ, as evidenced by the countable density in ordinal derivatives (Bessaga & Pełczyński, 1960).

Proof. From the induction in Lemma 3.1, the operator $\mathcal{S}_n$ at each level's factors through C_0 by Lemma 4.6 of Acuaviva (2026). The transfinite union is countable, so S factors through $C_0(N)$, which is isomorphic to C_0(Pełczyński, 1960).

3.2 Classification of Complemented Subspaces

Let

$$\mathcal{P}: C_0(\omega^\omega\times\mathcal{K}_{\mathcal{A}})\to C_0(\omega^\omega\times\mathcal{K}_{\mathcal{A}})$$

be a projection with infinite-dimensional image. From Lemma 3.1, we obtain $\mathcal{P}=Q\ \widehat{\otimes} I+\mathcal{S}$, with Q:$C(\omega^\omega)\to C(\omega^\omega)$ and $\mathcal{S}$ separable-range.

By (Acuaviva, 2026, Lemma 4.7) (extended analogously via induction), Q is a projection.

If $Q[C(\omega^\omega)]$ is finite-dimensional of dimension d, then $\mathcal{P}$ factors through $c_0(\mathcal{K}_{\mathcal{A}})^{d+1}$ by restricting to the finite ordinal part. By Acuaviva (2026, Proposition 4.9) (inductively extended), the image is isomorphic to c_0 or $c_0(\mathcal{K}_{\mathcal{A}})^\omega$ for $m\leq d+1$, using the factorization result in Lemma 4.8 of Laustsen (2002).

If $Q[C(\omega^\omega)]\sim C(\omega^\omega)$), conjugate to assume Q=I, so $\mathcal{P}$=I-$\mathcal{S}$. Then I-$\mathcal{P}$=$\mathcal{S}$, and $\mathcal{S}$ factors through c_0(Acuaviva, 2026, Lemma 4.6). By Lemma 4.8, $\mathcal{S}[C_0(\omega^\omega\times\mathcal{K}_{\mathcal{A}})]$ is finite dimensional or isomorphic to c_0. The image of P contains a complemented copy of c_0(Lotz et al., 1979) for scattered spaces. Thus,

$$C_0(\omega^\omega\times\mathcal{K}_{\mathcal{A}})\sim\mathcal{P}[\cdot]\oplus\mathcal{S}[\cdot]\sim\mathcal{P}[\cdot],$$

as the finite or C_0 summand is absorbed isomorphically in infinite-dimensional scattered function spaces (Bessaga & Pełczyński, 1960).

Proof of Theorem 1.1

Let $\mathcal{P}$: $C_0(\omega^\omega\times\mathcal{K}_{\mathcal{A}})\to C_0(\omega^\omega\times\mathcal{K}_{\mathcal{A}})$ be a projection with infinite-dimensional image. By Lemma 3.1, $\mathcal{P}=Q\ \widehat{\otimes} I+\mathcal{S}$, , where $Q\in B(C(\omega^\omega))$ and $\mathcal{S}$ has separable range. Since $\mathcal{P}^2=P$, we compute:

$$\mathcal{P}^2=(Q^2\ \widehat{\otimes} I)+(Q\ \widehat{\otimes} I)\mathcal{S}+\mathcal{S}(Q\ \widehat{\otimes} I)+\mathcal{S}^2.$$

Subtracting P from both sides yields:

$$0 = \mathcal{P}^2 - \mathcal{P} = (Q^2 - Q) \widehat{\otimes} I + [(Q \widehat{\otimes} I)\mathcal{S} + \mathcal{S}(Q \widehat{\otimes} I) + \mathcal{S}^2 - \mathcal{S}].$$

The term in brackets has separable range, being a finite sum of compositions involving $\mathcal{S}$. Consequently, (Q^2-Q) $\widehat{\otimes}$I has separable range. Suppose Q^2-Q≠0. Then there exists $f \in C(\omega^\omega)$ such that g=(Q^2-Q)f≠0. The range of (Q^2-Q) $\widehat{\otimes}$I contains all elementary tensors g⊗h for h∈$C_0(\mathcal{K}_{\mathcal{A}})$. Since $\mathcal{K}_{\mathcal{A}}$ is generated by an almost disjoint family of continuum cardinality, $C_0(\mathcal{K}_{\mathcal{A}})$ is non-separable. Thus, the set $\{g \otimes h : h \in B_{C0(\mathcal{K}\mathcal{A})}\}$ is non-separable in $C(\omega^\omega) \widehat{\otimes} C_0(\mathcal{K}_{\mathcal{A}})$, contradicting the separability of ran ((Q^2-Q)$\widehat{\otimes}$I). Therefore, Q^2-Q=0, and Q is a projection on $C(\omega^\omega)$.

- **Case 1:** ran(Q) is finite-dimensional. Let dim(ran(Q))=d<∞. Then Q factors through $\mathbb{k}^d$, and $\mathcal{P}$=(Q$\widehat{\otimes}I$)+$\mathcal{S}$ factors through $C_0(\mathcal{K}_{\mathcal{A}})^d \oplus Z$ where Z is a separable space. Hence ran($\mathcal{P}$) is a complemented subspace of a finite ℓ_∞-sum of copies of $C_0(\mathcal{K}_{\mathcal{A}})$ and a separable space. By the classification of complemented subspaces in spaces with few operators (Koszmider & Laustsen, 2021), any infinite-dimensional complemented subspace of such a sum is isomorphic to c_0 or $C_0(\mathcal{K}_{\mathcal{A}})^m$ for some $m \in \mathbb{N}$
- **Case 2:** ran(Q) is infinite-dimensional. Since Q is a projection on $C(\omega^\omega)$ and ω^ω is a countable compact metric space, the classification of complemented subspaces of $C(\mathcal{K})$ (Bessaga & Pełczyński, 1960) implies ran(Q)≅$C(\omega^\omega)$. Decompose the space as $C(\omega^\omega \times \mathcal{K}_{\mathcal{A}}) = \mathcal{X} \oplus \mathcal{Y}$, where $\mathcal{X}$=ran($\mathcal{P}$) and $\mathcal{Y}$=ker($\mathcal{P}$). The operator $\mathcal{P}$ coincides with Q$\widehat{\otimes}$I modulo the separable-range perturbation $\mathcal{S}$. Consequently, $\mathcal{X}$ contains a complemented copy of ran(Q) $\widehat{\otimes}_\varepsilon$ $C_0(\mathcal{K}_{\mathcal{A}}) \cong C(\omega^\omega)$ $\widehat{\otimes}_\varepsilon$ $C_0(\mathcal{K}_{\mathcal{A}}) \cong C_0(\omega^\omega \times \mathcal{K}_{\mathcal{A}})$. Thus, there exists a bounded projection from $\mathcal{X}$ onto a subspace isomorphic to $C_0(\omega^\omega \times \mathcal{K}_{\mathcal{A}})$.

We now examine the structure of $\mathcal{Y}$=ker($\mathcal{P}$)=ran(I-$\mathcal{P}$). From $\mathcal{P}^2$=$\mathcal{P}$, we have I-$\mathcal{P}$= -(S+(I-$\mathcal{P}$)$\mathcal{S}$). Since $\mathcal{S}$ has separable range, I-$\mathcal{P}$ is a sum of separable-range operators, hence Y is a separable complemented subspace of $C_0(\omega^\omega \times \mathcal{K}_{\mathcal{A}})$.

Since $C_0(\omega^\omega \times \mathcal{K}_{\mathcal{A}}) = \mathcal{X} \oplus \mathcal{Y}$ and $\mathcal{X}$ contains a complemented subspace isomorphic to $C_0(\omega^\omega \times \mathcal{K}_{\mathcal{A}})$, the mutual complemented embeddings allow an application of the Pełczyński decomposition method (Pełczyński, 1960), yielding $\mathcal{X} \cong C_0(\omega^\omega \times \mathcal{K}_{\mathcal{A}})$.

Combining Case 1 and Case 2, every infinite-dimensional complemented subspace of $C_0(\omega^\omega \times \mathcal{K}_{\mathcal{A}})$is isomorphic to c_0, $C_0(\mathcal{K}_{\mathcal{A}})^n$ for some $n \in \mathbb{N}$, or $C_0(\omega^\omega \times \mathcal{K}_{\mathcal{A}})$. The uniqueness of these isomorphism classes follows from the invariance of the Cantor-Bendixson index under Banach space isomorphism and the separability character of the respective quotient spaces.

Corollary 3.3

The classification holds for $C_0(\omega^\omega \times \mathcal{K}_{\mathcal{A}})$ for finite n, by iterated application of Lemma 3.1.

Proof. For n=1, the result reduces to (Acuaviva, 2026, Theorem 1.4). Inductively, assume for n-1. Then

$$C_0(\omega^n \times \mathcal{K}_{\mathcal{A}}) \cong C_0(\omega^{n-1} \times \mathcal{K}_{\mathcal{A}}) \oplus_\infty C_0(\omega \times \mathcal{K}_{\mathcal{A}}),$$

by ordinal decomposition (Bessaga & Pełczyński,1960). Projections decompose via Lemma 3.1, and complemented subspaces classify by adding the finite power cases from the induction hypothesis, using Peczyski decomposition for $\oplus_\infty$ sums (Pełczyński, 1960).

Remark 3.4. Unlike the separable case considered by (Benyamini, 1978), the non-separable factor introduces finite powers $C_0(\mathcal{K}_{\mathcal{A}})^n$, observed in (Candido, 2021) for specific ordinals under ♣-principle.

Proof. In pure separable $C(\omega^\omega)$, Benyamini,1978) shows no finite powers due to the ordinal structure preventing stable finite-dimensional complements. The tensor with few operator $C_0(\mathcal{K}_{\mathcal{A}})$ allows finite-rank perturbations to stabilize as powers, as seen in the finite-dimensional case of Lemma 3.1 and Theorem 1.1 of Candido (2021) for α=1.

4. Applications

4.1 Operator Ideals

We extend the structural ideal analysis initiated in Acuaviva (2026). Using the standardized notation introduced above, we describe the natural hierarchy of operator ideals associated with $C_0(\omega^\omega \times \mathcal{K}_{\mathcal{A}})$.

Theorem 4.1. The algebra $B(C_0(\omega^\omega \times \mathcal{K}_{\mathcal{A}}))$ admits the following natural chain of closed ideals:

$$\{0\} \subseteq \mathcal{K}(\mathcal{X}) \subseteq G_{c_0}(\mathcal{X}) \subseteq \mathcal{S}(\mathcal{X}) \subseteq B(\mathcal{X}).$$

where $\mathcal{X}$= $C_0(\omega^\omega \times \mathcal{K}_{\mathcal{A}})$ and all ideals are understood to act on $\mathcal{X}$.

Proof.

Since $\mathcal{X}$ is an $\mathcal{L}_\infty$-space with the bounded approximation property (Kania & Kochanek, 2014), the ideal $\mathcal{K}(\mathcal{X})$ coincides with the norm-closure of the finite rank operators. Every finite-rank operator factor through a finite-dimensional l_∞^n space, which embeds complementedly into c_0. By standard factorization theory, this implies $\mathcal{K}(\mathcal{X}) \subseteq G_{C0}(\mathcal{X})$.

By definition, any $\mathcal{T} \in G_{C0}(\mathcal{X})$ factors as $\mathcal{T}$=V∘W with $W: \mathcal{X} \to c_0$ and $V: c_0 \to \mathcal{X}$. Since c_0 is separable and bounded operators preserve separability of ranges, ran(V) is separable. Hence ran($\mathcal{T}$)⊆ran(V) is separable, which

yields $G_{C0}(\mathcal{X}) \subseteq \mathcal{S}(\mathcal{X})$. The inclusion $\mathcal{S}(\mathcal{X}) \subseteq B(\mathcal{X})$ is immediate by definition.

We emphasize that this chain reflects the structural hierarchy induced by standard factorization properties. Strictness of the inclusions, maximality of the ideals, and completeness of the lattice require further operator-algebraic analysis and are not claimed here.

4.2 Realization of Algebras

The structural hierarchy of ideals established in Theorem 4.1 suggests a natural connection to quotient algebras. Specifically, the separable-range ideal $\mathcal{S}(\mathcal{X})$ forms a closed two-sided ideal in $B(\mathcal{X})$, where $\mathcal{X}=C_0(\omega^\omega \times \mathcal{K}_{\mathcal{A}})$. By Lemma 3.1, the canonical quotient map $\pi: B(\mathcal{X}) \to B(\mathcal{X})/\mathcal{S}(\mathcal{X})$ effectively isolates the tensor component $U \widehat{\otimes} I$, which acts on the non-separable factor $C(\omega^\omega)$.

Following the structural framework developed in Acuaviva (2026, Remark 1.8), this quotient algebra exhibits structural features analogous to those of $B(c_0)$. In particular, for a suitably chosen almost disjoint family B, the quotient $B(C_0(\omega^\omega \times \mathcal{K}_B))/\mathcal{S}(C_0(\omega^\omega \times \mathcal{K}_B))$ shares certain structural properties with $B(c_0)$.A complete realization theorem, establishing an explicit isometric or isomorphic quotient equivalence, requires a detailed analysis of the Calkin-type structure and the behavior of the projection map modulo separable perturbations. Such an analysis is beyond the scope of the present manuscript but would build upon the ideal decomposition provided in Section 4.1 and the transfinite induction framework of Lemma 3.1.

Remark 4.2. The classification framework discussed above can be adapted to disjoint unions of the form $c_0((\omega^\omega \times \mathcal{K}_{\mathcal{A}}) \sqcup L) \cong c_0(\omega^\omega \times \mathcal{K}_{\mathcal{A}}) \oplus_\infty c_0(L)$. When $c_0(L)$ is a Grothendieck space, the ideal structure of the direct sum exhibits analogous complementation phenomena to those established in Theorem 1.1, in the spirit of the operator extension techniques developed by Johnson et al. (2016).

5. Discussion

The results extend Acuaviva's classifications (Acuaviva, 2026) to transfinite ordinals, leveraging Benyamini's extension theorem (Benyamini,1978) and tensor decompositions (Ryan, 2002). Compared to (Benyamini,1978), which classifies complemented subspaces of $C(\omega^\omega)$ as c_0 or the full space using extension properties for separable spaces, our work incorporates non-separable factors with few operators, introducing finite powers $C_0(\mathcal{K}_{\mathcal{A}})^n$.

This novelty addresses the gap in (Acuaviva, 2026) for higher ordinals, without set-theoretic assumptions like CH or the ♣-principle required in earlier works (Candido, 2021; Koszmider, 2005; Koszmider & Zieliński, 2011). Evidence from Bessaga and Pełczyński (1960) supports the ordinal induction, as the function algebra of scattered spaces are determined by Cantor-Bendixson indices.

Recent extensions by Candido (2004, 2021) to $C_0(\alpha \times L)$ and $C_0(L \times L)$ confirm similar dichotomies, but our approach generalizes to ω^ω via uniform crossnorms, highlighting the role of injective tensors in preserving complements (Ryan, 2002). Unlike Koszmider and Laustsen (2021), which provides a classification for $C_0(\mathcal{K}_{\mathcal{A}})$, our tensorial method yields ideal lattices akin to $B(c_0)$ quotients (Gohberg et al.,1967), with applications to unique algebra norms (Arnott and Laustsen, 2024).

Limitations include the focus on scattered spaces; non-scattered extensions remain open, as in (Plebanek et al.,2026). Remarks like Corollary 3.3 suggest scalability, supported by Milutin's universality (Milutin,1966).

6. Conclusion

This paper classifies complemented subspaces of $C_0(\omega^\omega \times \mathcal{K}_{\mathcal{A}})$ as $c_0, C_0(\mathcal{K}_{\mathcal{A}})^n$, or the full space, and the results extend those of Acuaviva (2026) by means of Benyamini's extension theorem (Benyamini, 1978), and tensor theory (Ryan, 2002). Key contributions include transfinite ordinal handling and ideal lattice descriptions, supported by inductive proofs and citations (Bessaga & Pełczyński, 1960; Gohberg et al., 1967).

Future work should generalize to arbitrary compact metric spaces M (Candido, 2021) and non-BAP cases (Ryan, 2002), potentially using slice maps as noted in (Acuaviva, 2026).

7. References

1. *Acuaviva, A. (2026). Operators on injective tensor products of separable Banach spaces and spaces with few operators. Journal of Functional Analysis, 290, 111203. https://doi.org/10.1016/j.jfa.2025.111203*
2. *Arnott, M., & Laustsen, N. J. (2024). Uniqueness of algebra norm on quotients of the algebra of bounded operators on a Banach space. Journal of Functional Analysis, 287, 110559. https://doi.org/10.1016/j.jfa.2024.110559*
3. *Benyamini, Y. (1978). An extension theorem for separable Banach spaces. Israel Journal of Mathematics, 29, 53-62. https://doi.org/10.1007/BF02760399*
4. *Bessaga, C., & Pełczyński, A. (1960). Spaces of continuous functions (IV): On isomorphical classification of spaces of continuous functions. Studia Mathematica, 19, 53-62.*

5. *Candido, L. (2021). On Banach spaces of the form $C_0(\alpha \times L)$ with few operators. Banach Journal of Mathematical Analysis, 15, 1-23. https://doi.org/10.1007/s43037-021-00126-w*
6. *Candido, L. (2025). Few operators on Banach spaces $C_0(L \times L)$. Topology and Its Applications, 375, 109577. https://doi.org/10.1016/j.topol.2025.109577*
7. *Gohberg, I. C., Markus, A. S., & Feldman, I. A. (1967). Normally solvable operators and ideals associated with them. American Mathematical Society Translations, 61, 63-84.*
8. *Hjorth, G. (2005). T. Jech. Review of Set theory: The third millennium edition, revised and expanded. Bulletin of Symbolic Logic, 11(2), 243-245. https://doi.org/10.1017/S1079898600003358*
9. *Johnson, W. B., Kania, T., & Schechtman, G. (2016). Closed ideals of operators on and complemented subspaces of Banach spaces of functions with countable support. Proceedings of the American Mathematical Society, 144, 4471-4485. https://doi.org/10.1090/proc/13084*
10. *Kania, T., & Kochanek, T. (2014). The ideal of weakly compactly generated operators acting on a Banach space. Journal of Operator Theory, 71, 455-477. https://doi.org/10.7900/jot.2012jun23.1959*
11. *Koszmider, P. (2005). On decompositions of Banach spaces of continuous functions on Mrówka's spaces. Proceedings of the American Mathematical Society, 133, 2137-2146. https://doi.org/10.1090/S0002-9939-05-07799-3*
12. *Koszmider, P., & Laustsen, N. J. (2021). A Banach space induced by an almost disjoint family admitting only few operators and decompositions. Advances in Mathematics, 381, 107613. https://doi.org/10.1016/j.aim.2021.107613*
13. *Koszmider, P., & Zieliński, P. (2011). Complementation and decompositions in some weakly Lindelöf Banach spaces. Journal of Mathematical Analysis and Applications, 376, 329-341. https://doi.org/10.1016/j.jmaa.2010.11.014*
14. *Kunen, K. (2004). Compact scattered spaces in forcing extensions. arXiv. https://doi.org/10.48550/arXiv.math/0408039*
15. *Laustsen, N. J. (2002). Maximal ideals in the algebra of operators on certain Banach spaces. Proceedings of the Edinburgh Mathematical Society, 45, 523-546. https://doi.org/10.1017/S0013091500001097*
16. *Lotz, H. P., Peck, N. T., & Porta, H. (1979). Semi-embeddings of Banach spaces. Proceedings of the Edinburgh Mathematical Society, 22, 233-240. https://doi.org/10.1017/S0013091500016394*
17. *Milutin, A. A. (1966). Isomorphism of the spaces of continuous functions over compacta of the cardinality of the continuum. Teor. Funkcii Funkcional. Anal. i Priložen., 2, 150-156.*
18. *Pełczyński, A. (1960). Projections in certain Banach spaces. Studia Mathematica, 19, 209-228.*
19. *Plebanek, G., Rondoš, J., & Sobota, D. (2026). Complemented subspaces of Banach spaces $C(K \times L)$. Journal of Functional Analysis, 290, 111236. https://doi.org/10.1016/j.jfa.2025.111236*
20. *Ryan, R. A. (2002). Introduction to tensor products of Banach spaces. Springer. https://doi.org/10.1007/978-1-4471-3903-4*
21. *Shelah, S. (1978). A Banach space with few operators. Israel Journal of Mathematics, 30, 181-191. https://doi.org/10.1007/BF02760838*
22. *Shelah, S., & Steprāns, J. (1988). A Banach space on which there are few operators. Proceedings of the American Mathematical Society, 104, 101-105. https://doi.org/10.1090/S0002-9939-1988-0958051-9*
23. *Wark, H. M. (2001). A non-separable reflexive Banach space on which there are few operators. Journal of the London Mathematical Society, 64, 675-689. https://doi.org/10.1112/S0024610701002393*