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Weighted Composition Operators on Bloch-Type Spaces in Infinite-Dimensional Hilbert Spaces: Norms and Dynamics

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Abstract
We study weighted composition operators (WCO) on Bloch-type spaces (BTS) defined over bounded domains in infinite-dimensional complex Hilbert spaces, extending classical results from the unit disk and finite-dimensional unit ball. Necessary and sufficient conditions are obtained for their boundedness and compactness in terms of the inducing map and the weight function. Explicit estimates for the operator norm and asymptotic formulas for the essential norm are derived via boundary limsup expressions. Finally, we analyze the dynamical behavior of iterations and establish conditions ensuring convergence to zero in the operator norm. These results unify and generalize known theorems in both finite- and infinite-dimensional settings and provide a framework for further investigations.

Keywords: Bloch-type spaces; weighted composition operators; Hilbert spaces; essential norm; compact operators; operator dynamics

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1. Introduction

Composition operators and (WCO) have been extensively studied on classical spaces of analytic functions such as Hardy (see [1-3]), Bergman (see [4-5]), and Bloch spaces (see [6-9]). Bloch spaces have attracted considerable attention due to their close connection with hyperbolic geometry and operator theory. In several complex variables, (BTS) have been investigated on domains such as the unit ball and the polydisk, where boundedness and compactness of (WCO)are typically characterized via boundary behavior and derivative growth conditions (see [10-14, 15-19]). In the infinite-dimensional setting, [20, 21] introduced BTS on the unit ball of a complex Hilbert space and established foundational results. Additional developments can be found in references [22-25]. Recently, there has been growing interest in (WCO)between BTS and other function spaces in Banach and Hilbert spaces (see [26-28, 16-18]). However, a unified treatment addressing boundedness, compactness, essential norm estimates, and dynamical behavior simultaneously in both finite- and infinite-dimensional settings is still lacking. The aim of this paper is to provide such a framework. The organization of this paper is as follows: Section 1 is the introduction to the investigation. Preliminaries are given in Section 2. The main results are described in Section 3. Finally, Section 4 is conclusion of the paper.

2. Preliminaries

- **Definition (2.1):**

Let H be a complex Hilbert space, and let D⊂H be a bounded domain. The main example to keep in mind is the open unit ball.

$$\boldsymbol{B_H} = \{\boldsymbol{z} \in \boldsymbol{H}: \| \boldsymbol{z} \| < 1\} \tag{2.1}$$

Denote by H(D) the space of all holomorphic (Fréchet-analytic) functions f:D→C. For f∈H(D)and z∈D, we write Df(z)for the Fréchet derivative of fat z, which is a bounded linear functional on H; its operator norm is denoted by ‖Df(z)‖.

- **Definition (2.2):**

The Bloch space B(D) consists of all holomorphic functions f∈H(D)such that

$$\boldsymbol{\beta(f)} = \sup_{\boldsymbol{z}\in\boldsymbol{D}} \boldsymbol{\delta(z)} \parallel \boldsymbol{Df(z)} \parallel < \infty \tag{2.2}$$

where δ(z) is a fixed boundary distance function on D(for instance, δ(z)=1-‖z‖ when D=B_H). The Bloch norm is defined by

$$\| \boldsymbol{f} \|_{\boldsymbol{\mathcal{B}(D)}} = | \boldsymbol{f(0)} | + \boldsymbol{\beta(f)} \tag{2.3}$$

Remark. The quantity β(f) is the Bloch seminorm and measures the growth of f near the boundary ∂D.

- **Definition (2.3):**

The little Bloch space B_0(D) is defined by

$$\boldsymbol{B_0(D)} = \{\boldsymbol{f} \in \boldsymbol{\mathcal{B}(D)}: \boldsymbol{\delta(z)} \parallel \boldsymbol{Df(z)} \parallel \to \boldsymbol{0} \text{ as } \boldsymbol{z} \to \boldsymbol{\partial D}\} \tag{2.4}$$

- **Definition (2.4):**

To work in a more general setting, we introduce BTS defined via a continuous weight function.

Let μ:D→(0,∞)be a continuous weight. We define the Bloch-type space $B_μ$(D)be a continuous weight. We define the Bloch-type space by

$$\mathbf{B}_{\boldsymbol{\mu}}(\boldsymbol{D}) = \left\{\boldsymbol{f} \in \boldsymbol{H(D)}: \| \boldsymbol{f} \|_{\boldsymbol{\mathcal{B}_\mu(D)}} := | \boldsymbol{f(0)} | + \sup_{\boldsymbol{z}\in\boldsymbol{D}} \boldsymbol{\mu(z)} \parallel \boldsymbol{Df(z)} \parallel < \infty\right\} \tag{2.5}$$

We assume throughout that μ is a normal weight in the sense that:

$$\boldsymbol{\mu(z)} \to \boldsymbol{0} \text{as } \boldsymbol{z} \to \boldsymbol{\partial D} \tag{2.6}$$

μ(z) is comparable to a boundary distance function; for example, on the Hilbert ball B_H,

$$\boldsymbol{c_1}(\boldsymbol{1} - \| \boldsymbol{z} \|) \le \boldsymbol{\mu(z)} \le \boldsymbol{c_2}(\boldsymbol{1} - \| \boldsymbol{z} \|) \tag{2.7}$$

for some constants c_1,c_2>0.

The associated little Bloch-type space is

$$\boldsymbol{\mathcal{B}_{\mu,0}(D)} = \{\boldsymbol{f} \in \boldsymbol{\mathcal{B}_\mu(D)}: \boldsymbol{\mu(z)} \parallel \boldsymbol{Df(z)} \parallel \to \boldsymbol{0} \text{ as } \boldsymbol{z} \to \boldsymbol{\partial D}\}.$$

3. Main Results

In this section, we study WCO on BTS in a Hilbert space.

Let D⊂H be a bounded domain in a complex Hilbert space H. Let φ:D→D be an analytic self-map (the symbol of the composition operator) and let ψ∈H(D) be an analytic weight function. We define the weighted composition operator

$$W_{\psi,\varphi}: H(D) \to H(D), W_{\psi,\varphi} f = \psi \cdot (f \circ \varphi) \tag{3.1}$$

This boundedness criterion extends and unifies several known results on WCO between BTS in the disk, the unit ball, and more general Banach-space balls; see [10, 11, 13, 14, 16-19, 25-28].

3.1 Theorem (Boundedness)

Let D be a bounded domain in a complex Banach space (in particular, a complex Hilbert space), and let $B_μ$ (D)be the BTS associated to a normal weight μ. Let φ:D→D and ψ∈H(D) be analytic Assume that there exists a family of test functions $\{f_a\}_{a\in D}$⊂$B_μ$(D) satisfying.

$$\sup_{a\in D} \| f_a \|_{B_{\mu(D)}} < \infty, \tag{T1}$$

$$\mu(a) \| Df_a(a) \| \ge c_0 > 0, \tag{T2}$$

and

$f_a \to 0$ locally uniformly as $a \to \partial D$ (T3)

Then the WCO $W_{\psi,\varphi}: B_\mu(D) \to B_\mu(D)$ is bounded if and only if the following two conditions hold:

$$\sup_{z\in D} \mu(z) \, \| D\psi(z) \| < \infty \quad (C1) \tag{3.2}$$

$$\sup_{z\in D} \frac{\mu(z)}{\mu(\varphi(z))} \, |\psi(z)| \, \| D\varphi(z) \| < \infty \quad (C2) \tag{3.3}$$

Proof.

Assume first that conditions (C1) and (C2) hold. Let $f \in B_\mu(D)$. By the product rule and the chain rule for Fréchet derivatives,

$$D(W_{\psi,\varphi} f)(z) = D\psi(z)\, f(\varphi(z)) + \psi(z)\, (Df)(\varphi(z)) \circ D\varphi(z). \tag{3.4}$$

Taking norms and multiplying by $\mu(z)$, we get

$$\mu(z) \, \|D(W_{\psi,\varphi} f)(z)\| \le \mu(z) \, \|D\psi(z)\| \, |f(\varphi(z))| + \mu(z) \, |\psi(z)| \, \|D\varphi(z)\| \, \|Df(\varphi(z))\|. \tag{3.5}$$

We now estimate the first term.Since point evaluations are bounded on for each $w \in D$ the evaluation functional $\delta_w(f) = f(w)$ is bounded. Hence there exists a constant $C_1 > 0$, depending only on D and μ, such that

$$| f(\varphi(z)) | \le C_1 \, \| f \|_{B_\mu(D)}, z \in D. \tag{3,6}$$

Therefore,

$$\mu(z) \, \|D\psi(z)\| \, |f(\varphi(z))| \le C_1 \, \mu(z) \, \|D\psi(z)\| \, \|f\|_{B_\mu(D)}. \tag{3,7}$$

For the second term, the definition of the Bloch-type norm implies

$$\mu(\varphi(z)) \, \|Df(\varphi(z))\| \le \|f\|_{B_\mu(D)},$$

Thus,

$$\mu(z) \, |\psi(z)| \, \|D\varphi(z)\| \, \|Df(\varphi(z))\| \le \frac{\mu(z)}{\mu(\varphi(z))} \, |\psi(z)| \, \|D\varphi(z)\| \, \|f\|_{B_\mu(D)}.$$

Combining this with (3.5), we obtain

$$\mu(z) \, \|D(W_{\psi,\varphi} f)(z)\| \le \left(C_1 \, \mu(z) \, \|D\psi(z)\| + \frac{\mu(z)}{\mu(\varphi(z))} \, |\psi(z)| \, \|D\varphi(z)\| \right) \|f\|_{B_\mu(D)}.$$

Taking the supremum over $z \in D$ and using assumptions (C1) and (C2) yields

$$\sup_{z\in D} \mu(z) \, \|D(W_{\psi,\varphi} f)(z)\| \le C \, \|f\|_{B_\mu(D)} \tag{3,8}$$

for some constant $C > 0$. Moreover, The zeroth-order term satisfies

$$|W_{\psi,\varphi} f(0)| = |\psi(0)| \, |f(\varphi(0))| \le C' \|f\|_{B_\mu(D)}$$

Hence

$$\|W_{\psi,\varphi} f\|_{B_\mu(D)} \le C'' \|f\|_{B_\mu(D)}$$

so $W_{\psi,\varphi}$ is bounded.

Now assume that $W_{\psi,\varphi}$ is bounded on $B_\mu(D)$. We prove the necessity of (C1) and (C2).

First, taking $f \equiv 1$, we have

$$W_{\psi,\varphi}(1) = \psi.$$

Since $W_{\psi,\varphi}$ maps $B_\mu(D)$ into itself, it follows that $\psi \in B_\mu(D)$, and therefore

$$\sup_{z\in D} \mu(z) \, \|D\psi(z)\| < \infty.$$

This gives (C1).

To prove (C2), let $z \in D$ and set $a = \varphi(z)$. By the test-function hypothesis, choose $f_a \in B_\mu(D)$ satisfying (T1)-(T3). Since $W_{\psi,\varphi}$ is bounded,

$$\mu(z) \, \| D(W_{\psi,\varphi} f_a)(z) \| \le C.$$

Using

$$D(W_{\psi,\varphi} f_a)(z) = D\psi(z)\, f_a(\varphi(z)) + \psi(z)\, (Df_a)(\varphi(z)) \circ D\varphi(z),$$

we obtain

$$\mu(z) \, | \psi(z) | \, \| (Df_a)(\varphi(z)) \circ D\varphi(z) \| \le C + \mu(z) \, \| D\psi(z) \| \, | f_a(\varphi(z)) |.$$

By (C1) and boundedness of point evaluations, the right-hand side is uniformly bounded. Since $a = \varphi(z)$, condition (T2) gives

$$\mu(\varphi(z)) \, \| Df_{\varphi(z)}(\varphi(z)) \| \ge c_0.$$

Hence

$$\frac{\mu(z)}{\mu(\varphi(z))} \, | \psi(z) | \, \| D\varphi(z) \| \le C',$$

uniformly in $z \in D$. Thus (C2) holds.

Therefore, $W_{\psi,\varphi}$ is bounded on $B_\mu(D)$ if and only if (C1) and (C2) hold.

3.2 Theorem (Compactness)

With the same assumptions and notations as in Theorem 3.1, consider the little Bloch-type space $B_{\mu,0}(D)$. Then $W_{\psi,\varphi}: B_{\mu,0}(D) \to B_{\mu,0}(D)$ is compact if and only if

$$\lim_{z\to\partial D} \mu(z) \, \| D\psi(z) \| = 0 \tag{3.9}$$

$$\lim_{z\to\partial D} \frac{\mu(z)}{\mu(\varphi(z))} \, \| \psi(z) D\varphi(z) \| = 0 \tag{3.10}$$

Proof

Assume first that $W_{\psi,\varphi}$ is compact. Let $(z_n)\subset D$ be any sequence with $z_n\to\partial D$. Since μ is a normal weight, we can choose a bounded sequence $(f_n)\subset B_{\mu,0}(D)$ such that $f_n\to 0$ locally uniformly in D and

$$\|f_n\|_{B_\mu(D)} \le 1, \mu(z_n)\|Df_n(z_n)\| \ge c > 0$$

for some constant $c>0$. Compactness implies

$$\|W_{\psi,\varphi}f_n\|_{B_\mu(D)} \to 0.$$

Using

$$D(W\psi,\varphi fn)(z) = D\psi(z)fn(\varphi(z)) + \psi(z)(Dfn)(\varphi(z))\circ D\varphi(z)$$

and evaluating near the boundary, we obtain that both boundary quantities in (3.9) and (3.10) must vanish; otherwise the derivative norm of $W_{\psi,\varphi}$ f_n would not tend to zero.

Conversely, assume (3.9) and (3.10).

By the standard sequential characterization of compact operators on $B_{\mu,0}(D)$ it suffices to show that every bounded sequence in $B_{\mu,0}(D)$ converging to zero locally uniformly is mapped to a sequence converging to zero in norm.

Let (f) be a bounded sequence in $B_{\mu,0}(D)$ converging locally uniformly to 0

On the compact set D \ U local uniform convergence implies uniform convergence of f_n to zero. By Cauchy estimates for Fréchet-holomorphic functions, the derivatives Df_n also converge uniformly to zero on compact subsets of D

Let (f_n)be a bounded sequence in $B_{(\mu,0)}(D)$ converging locally uniformly to. Fix $\varepsilon>0$. Since (3.9) and (3.10) hold, there exists a neighborhood U of ∂D such that for all $z\in U$,

$$\mu(z)\|D\psi(z)\| < \varepsilon, \frac{\mu(z)}{\mu(\varphi(z))}|\psi(z)|\|D\varphi(z)\| < \varepsilon.$$

On the compact set , local uniform convergence implies that and uniformly. Hence the derivative formula yields

$$\|W_{\psi,\varphi}f_n\|_{B_\mu(D)} \to 0.$$

Therefore $W_{\psi,\varphi}$ is compact

3.3 Proposition (Operator norm estimate

Under the assumptions of Theorem 3.1, there exists a constant $C>0$, depending only on the geometry of D and the weight μ, such that

$$\| W_{\psi,\varphi} \| \le C(|\psi(0)| + \sup_{z\in D}\mu(z) \| D\psi(z) \| + \sup_{z\in D}\frac{\mu(z)}{\mu(\varphi(z))} \| \psi(z)\, D\varphi(z) \|) \tag{3.11}$$

Proof

Let $f\in B_\mu$ (D). From the proof of Theorem 3.1 we already have

$$|W_{\psi,\varphi}f(0)| \le C_0|\psi(0)|\|f\|_{B_\mu(D)}$$

and

$$\sup_{z\in D}\mu(z)\|D(W_{\psi,\varphi}f)(z)\| \le C_1\left(\sup_{z\in D}\mu(z)\|D\psi(z)\| + \sup_{z\in D}\frac{\mu(z)}{\mu(\varphi(z))}|\psi(z)|\|D\varphi(z)\|\right)\|f\|_{B_\mu(D)}$$

Combining these estimates gives

$$\|W_{\psi,\varphi}f\|_{B_\mu(D)} \le C\left(|\psi(0)| + \sup_{z\in D}\mu(z)\|D\psi(z)\| + \sup_{z\in D}\frac{\mu(z)}{\mu(\varphi(z))}|\psi(z)|\|D\varphi(z)\|\right)\|f\|_{B_\mu(D)}.$$

Taking the supremum over all $\|f\|_{B\mu(D)}\le 1$ yields (3.11).

The dependence of comes only from the boundedness of point evaluations on $B_\mu(D)$ and from the comparability constants in the normality of μ .

3.4 Proposition (Essential norm estimate)

Let D,μ,ψ,φ be as above and assume that $W_{\psi,\varphi}$ is bounded on $B_\mu(D)$. Then the essential norm of $W_{\psi,\varphi}$ satisfies

$$\| W_{\psi,\varphi} \|_e \asymp \limsup_{z\to\partial D}\mu(z) \| D\psi(z) \| + \limsup_{z\to\partial D}\frac{\mu(z)}{\mu(\varphi(z))} \| \psi(z)\, D\varphi(z) \| \tag{3.12}$$

In particular, for the unit ball of a Hilbert space equipped with the standard Bloch weight, the essential norm is governed entirely by the boundary behavior of ψ and φ.

Here, the notation $\asymp$ means that there exist positive constants c and C, depending only on the domain D and the normal weight μ, c $A\le\|W_{\psi,\varphi}\|_e\le C$ A,

Proof

For the lower bound, let K be any compact operator on $B_\mu(D)$. Choose a bounded sequence $(f_n)\subset B_{\mu,0}(D)$ converging locally uniformly to 0, with derivatives concentrating near boundary points $z_n\to\partial D$. Since $Kf_n\to 0$, we get

$$\| W_{\psi,\varphi} - K \| \ge \limsup_{n\to\infty} \| (W_{\psi,\varphi} - K)f_n \| = \limsup_{n\to\infty} \| W_{\psi,\varphi}f_n \|.$$

By the derivative formula, the right-hand side dominates the boundary limsup terms in (3.12). This yields the lower estimate.

For the upper bound, we construct compact approximants K_r by cutting off the operator

near the boundary. Let χ_r be a smooth radial cutoff satisfying $\chi_r(z)=1$ on a smaller subdomain and $\chi_r(z)=0$ near ∂D. Define

$$K_r f := \chi_r \cdot W_{\psi,\varphi} f.$$

Since the support of χ_r is contained in a compact subset of D, the operator K_r f depends only on the behavior of $W_{\psi,\varphi}$ faway from the boundary. Moreover, bounded subsets of $B_\mu(D)$ are relatively compact in the topology of local uniform convergence on compact subsets. Therefore, K_r maps bounded sets into relatively compact sets and is thus a compact operator on $B_\mu(D)$. Similar arguments can be found in Zhao [10] and Wulan, Zheng and Zhu [11] for Bloch-type spaces.

Hence,

$$\| W_{\psi,\varphi} \|_e \leq \| W_{\psi,\varphi} - K_r \|.$$

Using the derivative estimate from Theorem 3.1 on the boundary layer where $\chi_r \neq 1$, we obtain

$$\| W_{\psi,\varphi} - K_r \| \lesssim \sup_{z \text{ near } \partial D} \left(\mu(z) \| D\psi(z) \| + \frac{\mu(z)}{\mu(\varphi(z))} | \psi(z) | \ \| D\varphi(z) \| \right).$$

Letting $r \to 1$ yields the upper estimate in (3.12).

3.5 Theorem (Dynamical behavior of iterates)

Let D be a bounded domain in a complex Banach space, let μ be a normal weight on D, and let $W_{\psi,\varphi}$ be a bounded weighted composition operator on $B_\mu(D)$ For each $n \in N$, define

$$\varphi^{\circ n} = \varphi \circ \cdots \circ \varphi. \tag{3.13}$$

$$\psi_n(z) = \prod_{k=0}^{n-1} \psi(\varphi^{\circ k}(z)), n \geq 1. \tag{3.14}$$

$$W^n_{\psi,\varphi} f = \psi_n \cdot (f \circ \varphi^{\circ n}) \tag{3.15}$$

Assume that φ has a Denjoy-Wolff point $\tau \in D$ and that

$$\lim_{n\to\infty} \sup_{z\in D} \mu(z) \| D\psi_n(z) \| = 0,$$

and

$$\lim_{n\to\infty} \sup_{z\in D} \frac{\mu(z)}{\mu(\varphi^{\circ n}(z))} | \psi_n(z) | \ \| D\varphi^{\circ n}(z) \| = 0. \tag{3.16}$$

Then

$$\| W^n_{\psi,\varphi} \| \longrightarrow 0 \text{as } n \to \infty.$$

Conversely, if

$$\| W^n_{\psi,\varphi} \| \longrightarrow 0,$$

then

$$\lim_{n\to\infty} \sup_{z\in D} \frac{\mu(z)}{\mu(\varphi^{\circ n}(z))} | \psi_n(z) | \ \| D\varphi^{\circ n}(z) \| = 0.$$

Proof

For each $n \in N$ and $f \in B_\mu(D)$,

$$W^n_{\psi,\varphi} f = \psi_n (f \circ \varphi^{\circ n}).$$

Differentiating, we obtain

$$D(W^n_{\psi,\varphi} f)(z) = D\psi_n(z) f(\varphi^{\circ n}(z)) + \psi_n(z) (Df)(\varphi^{\circ n}(z)) \circ D\varphi^{\circ n}(z).$$

Multiplying by μ(z) and taking norms yields

$$\mu(z) \| D(W^n_{\psi,\varphi} f)(z) \| \leq \mu(z) \| D\psi_n(z) \| \ | f(\varphi^{\circ n}(z)) | + \mu(z) | \psi_n(z) | \ \| D\varphi^{\circ n}(z) \| \| Df(\varphi^{\circ n}(z)) \|.$$

Since point evaluations are bounded on $B_\mu(D)$, there exists a constant $C_{ev} > 0$ such that

$$| f(\varphi^{\circ n}(z)) | \leq C_{\text{ev}} \| f \|_{B_\mu(D)}, z \in D.$$

Also, by the definition of the Bloch-type norm,

$$\mu(\varphi^{\circ n}(z)) \| Df(\varphi^{\circ n}(z)) \| \leq \| f \|_{B_\mu(D)}.$$

Hence,

$$\mu(z) \| D(W^n_{\psi,\varphi} f)(z) \| \leq \left(C_{\text{ev}} \mu(z) \| D\psi_n(z) \| + \frac{\mu(z)}{\mu(\varphi^{\circ n}(z))} | \psi_n(z) | \ \| D\varphi^{\circ n}(z) \| \right) \| f \|_{B_\mu(D)}.$$

Taking the supremum over $z \in D$, we get

$$\| W^n_{\psi,\varphi} f \|_{B_\mu(D)} \leq C \left(\sup_{z\in D} \mu(z) \| D\psi_n(z) \| + \sup_{z\in D} \frac{\mu(z)}{\mu(\varphi^{\circ n}(z))} | \psi_n(z) | \ \| D\varphi^{\circ n}(z) \| \right) \| f \|_{B_\mu(D)}.$$

Under the assumptions

$$\lim_{n\to\infty} \sup_{z\in D} \mu(z) | D\psi_n(z) | = 0 \text{ and } \lim_{n\to\infty} \sup_{z\in D} \frac{\mu(z)}{\mu(\varphi^{\circ n}(z))} | \psi_n(z) | \ \| D\varphi^{\circ n}(z) \| = 0,$$

the right-hand side tends to 0 as $n \to \infty$. Consequently,

$$\| W^n_{\psi,\varphi} \| \to 0.$$

Conversely, assume that $\|W^n_{\psi,\varphi}\| \to 0$. To recover the iterated symbol estimate, apply $W^n_{\psi,\varphi}$ to a family of test functions $\{f_a\}_{a\in D} \subset B_\mu(D)$ satisfying the same type of normalization used in Theorem 3.1, namely

$$\sup_{a\in D} \| f_a \|_{B_\mu(D)} < \infty, \mu(a) \| Df_a(a) \| \geq c_0 > 0.$$

Fix z∈D and set a=$\varphi^{\circ n}$ (z). Differentiating $W^n_{\psi,\varphi} f_a$ at z gives

$$D(W^n_{\psi,\varphi} f_a)(z) = D\psi_n(z)\, f_a(\varphi^{\circ n}(z)) + \psi_n(z)\, (Df_a)(\varphi^{\circ n}(z)) \circ D\varphi^{\circ n}(z).$$

Since $\|W^n_{\psi,\varphi}\| \to 0$, the left-hand side tends to 0 uniformly on bounded sets of test functions. Using the boundedness of point evaluations and the normalization

$$\mu(\varphi^{\circ n}(z)) \| Df_{\varphi^{\circ n}(z)}(\varphi^{\circ n}(z)) \| \geq c_0,$$

We obtain

$$\lim_{n\to\infty} \sup_{z\in D} \frac{\mu(z)}{\mu(\varphi^{\circ n}(z))} \, | \psi_n(z) | \, \| D\varphi^{\circ n}(z) \| \to 0.$$

uniformly in z∈D.

3.6 Corollary (Special cases and convergence)

In the setting of Theorem 3.5, suppose in addition that:

1. the Denjoy-Wolff point τ lies in the interior of D, and φ is a strict contraction in a suitable invariant metric near τ, so that $\|D\varphi^{\circ n}(z)\|$ decays at least geometrically along orbits
2. the weight ψ is continuous at τ and $|\psi(\tau)| < 1$, so that $|\psi_n(z)|$ decays at least geometrically as $n\to\infty$ for each z∈D.
3. the weight μ is normal, hence the ratio $\mu(z)/\mu(\varphi^{\circ n}(z))$ remains controlled along the orbits of φ.

Then

$$\limsup_{n\to\infty} \sup_{z\in D} \frac{\mu(z)}{\mu(\varphi^{\circ n}(z))} |\psi_n(z)| \, \|D\varphi^{\circ n}(z)\| = 0,$$

and consequently

$$\|W^n_{\psi,\varphi}\| \to 0$$

Proof

since φ is a strict contraction near its Denjoy-Wolff point, the derivative

$$\| D\varphi^{\circ n}(z) \|$$

decays geometrically along each orbit. The continuity of ψ at τ and the assumption

$$| \psi(\tau) | < 1$$

imply that $|\psi_n(z)|$ also decays geometrically.

Since the ratio

$$\frac{\mu(z)}{\mu(\varphi^{\circ n}(z))}$$

remains uniformly bounded along the orbits of φ , while both

$$| \psi_n(z) |$$

And $\| D\varphi^{\circ n}(z) \|$

decay geometrically, it follows that

$$\lim_{n\to\infty} \sup_{z\in D} \frac{\mu(z)}{\mu(\varphi^{\circ n}(z))} | \psi_n(z) | \| D\varphi^{\circ n}(z) \| = 0.$$

Hence, by Theorem 3.5,

$$\| W^n_{\psi,\varphi} \| \to 0.$$

3.7 Corollary (Power-compactness)

Under the assumptions of Theorem 3.5, if

$$\lim_{n\to\infty} \sup_{z\in D} \mu(z) \| D\psi_n(z) \| = 0$$

And

$$\lim_{n\to\infty} \sup_{z\in D} \frac{\mu(z)}{\mu(\varphi^{\circ n}(z))} | \psi_n(z) | \| D\varphi^{\circ n}(z) \| = 0.$$

Then

$$\| W^n_{\psi,\varphi} \| \to 0.$$

Consequently,

$$r_e(W_{\psi,\varphi}) = \lim_{n\to\infty} \| W^n_{\psi,\varphi} \|^{1/n}_e = 0 < 1.$$

Hence $W_{(\psi,\varphi)}$ is power-compact on B_μ (D).

By the assumptions,

$$\lim_{n\to\infty} \sup_{z\in D} \mu(z) \, | D\psi_n(z) | = 0,$$

and

$$\lim_{n\to\infty} \sup_{z\in D} \frac{\mu(z)}{\mu(\varphi^{\circ n}(z))} | \psi_n(z) | \, \| D\varphi^{\circ n}(z) \| = 0.$$

Therefore, all hypotheses of Theorem 3.5 are satisfied. It follows that

$$\| W^n_{\psi,\varphi} \| \longrightarrow 0 (n \to \infty).$$

Since the essential norm is dominated by the operator norm,

$$\| W^n_{\psi,\varphi} \|_e \leq \| W^n_{\psi,\varphi} \|,$$

we also obtain

$$\| W_{\psi,\varphi}^{n} \|_e \longrightarrow 0.$$

Hence,

$$r_e(W_{\psi,\varphi}) = \lim_{n\to\infty} \| W_{\psi,\varphi}^{n} \|_e^{1/n} = 0 < 1.$$

Since $r_e(W_{\psi,\varphi})<1$, it follows from the standard characterization of power-compact operators that $W_{\psi,\varphi}$ is power-compact on $B_\mu(D)$.

3.8 Example (A contractive diagonal map on the Hilbert ball)

Let H be a complex Hilbert space and let $B_H=\{z\in H:\|z\|<1\}$ be its open unit ball. Consider the standard Bloch-type space $B_\mu(B_H)$ associated with the normal weight

$$\mu(z) = 1 - \|z\|, z \in B_H.$$

Fix a constant λ with $0<|\lambda|<1$, and define the analytic self-map

$$\varphi(z) = \lambda z, z \in B_H. \qquad (3,17)$$

Let the weight be constant, $\psi(z)\equiv c\in C$. Then the weighted composition operator

$$W_{\psi,\varphi} f(z) = c\, f(\lambda z), f \in B_\mu(B_H)$$

is bounded on $B_\mu(B_H)$.

Indeed,

$$D\varphi(z) = \lambda I_H, D\psi(z) = 0. \qquad (3,18)$$

Hence condition (3.2) is immediate, and condition (3.3) becomes

$$\sup_{z\in B_H} \frac{1-\|z\|}{1-\|\lambda z\|} |c|\,|\lambda| < \infty,$$

which holds because

$$\frac{1-\|z\|}{1-\|\lambda z\|} = \frac{1-\|z\|}{1-|\lambda|\cdot\|z\|}$$

is bounded on B_H whenever $0<|\lambda|<1$.

For the iterates, we have

$$\varphi^{\circ n}(z) = \lambda^n z, \psi_n(z) = c^n, D\varphi^{\circ n}(z) = \lambda^n I_H.$$

Therefore,

$$\frac{\mu(z)}{\mu(\varphi^{\circ n}(z))} |\,\psi_n(z)\,|\,\|\,D\varphi^{\circ n}(z)\,\| = \frac{1-\|\,z\,\|}{1-|\,\lambda\,|^n\|\,z\,\|} |\,c\,|^n |\,\lambda\,|^n.$$

Since

$$\frac{1-\|z\|}{1-|\lambda|^n\|z\|} \le 1 \quad \text{for all } z \in B_H,$$

we obtain

$$\sup_{z\in B_H} \frac{\mu(z)}{\mu(\varphi^{\circ n}(z))} |\psi_n(z)|\,\|D\varphi^{\circ n}(z)\| \le (|c|\,|\lambda|)^n.$$

Consequently, if $|c\lambda|<1$, then the right-hand side tends to 0 as $n\to\infty$. By Theorem 3.5, it follows that

$$\|W_{\psi,\varphi}^{n}\| \to 0.$$

Hence $W_{\psi,\varphi}$ is power-compact, in agreement with Corollary 3.7.General diagonal operators in infinite-dimensional Hilbert spaces.

a diagonal map of the form

$$\varphi(z) = \sum_{j=1}^{\infty} \lambda_j \langle z, e_j\rangle e_j, 0 < \sup_j |\lambda_j| < 1,$$

with respect to an orthonormal basis $\{e_j\}$, and a constant weight $\psi\equiv c$. Then the same argument works with

$$\|D\varphi^{\circ n}(z)\| \le (\sup_j |\lambda_j|)^n$$

4. Conclusion

In this paper, we studied weighted composition operators acting on Bloch-type spaces defined on bounded domains in complex Hilbert spaces, using the standard operator . We established necessary and sufficient conditions for the boundedness of this operator on the Bloch space in terms of the analytic weight and the derivative of the inducing map , as reflected in the corresponding norm estimates, thereby extending related results in the literature.

In addition, we derived estimates for the operator norm and obtained a characterization of the essential norm through a boundary condition, which yields a compactness criterion and improves previously known essential norm estimates. Finally, we investigated the dynamical behavior of the iterates and gave conditions ensuring their convergence to zero in operator norm, formulated in terms of the weighted derivative growth along the orbits of , in the spirit of Denjoy-Wolff type dynamics for weighted composition operators.

These results extend several known theorems on composition and weighted composition operators to Bloch-type spaces over infinite-dimensional Hilbert domains and provide a solid foundation for further research in this direction.

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